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Construction of exact solutions to eigenvalue problems by the asymptotic iteration method

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Abstract

We apply the asymptotic iteration method (AIM) (Ciftci, Hall and Saad 2003 *J. Phys. A: Math. Gen.* **36** 11807) to solve new classes of second-order homogeneous linear differential equation. In particular, solutions are found for a general class of eigenvalue problems which includes Schrödinger problems with Coulomb, harmonic oscillator or Pöschl–Teller potentials, as well as the special eigenproblems studied recently by Bender *et al* (2001 *J. Phys. A: Math. Gen.* **34** 9835) and generalized in the present paper to arbitrary dimension.

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1. Introduction

The study of exactly solvable problems in quantum mechanics and the relation between their solutions represent valuable contributions to mathematical physics. The range of potentials for which Schrödinger's equation can be solved exactly has been extended considerably owing to the investigations inspired, for example, by super-symmetric quantum mechanics [1], shape invariance [2] and the factorization method [3–9]. Coulomb, harmonic oscillator and Pöschl–Teller potentials are known examples of exactly solvable problems. It is also known that Coulomb and harmonic oscillator potentials are inter-related and that they are members of a family of exactly solvable Schrödinger equations [10–12]. In this paper, we use the asymptotic iteration method (AIM) [13] to demonstrate that Pöschl–Teller potentials also belong to this family and that the (Coulomb, harmonic oscillator, Pöschl–Teller) exact results may be obtained by elementary transformations, as particular cases of the following second-order homogeneous linear differential equation:

$$y'' = 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(m+1)}{x}\right)y' - \frac{wx^N}{1-bx^{N+2}}y, \tag{1.1}$$

where $0 \leqslant x < b^{\frac{1}{N+2}}$ if $b > 0$ and $0 \leqslant x < \infty$ if $b \leqslant 0$. Here $N = -1, 0, 1, \ldots$, and the real numbers a, m and w are to be specified later.

We would like to make clear at the outset that all the problems discussed in this paper can be transformed into special cases of (1.1). The discovery of a common source for the entire set of problems and their treatment in arbitrary dimension d by the AIM are the most significant features of the paper. Our work has two related aspects: the solution of differential equations, and the solution of boundary-value problems. In particular, we shall show that exact solutions of the class of eigenvalue problem of the form

$$-y''(x) + x^{2N+2}y(x) = Ex^N y(x) \qquad (N = -1, 0, 1, \ldots) \tag{1.2}$$

recently studied by Bender *et al* [14] follow as special cases of the same differential equation (1.1). This will allow us to extend the solutions of (1.2) to provide analytic formulae for the solutions of the eigenvalue problem

$$-y''(x) + \left(\frac{m(m+1)}{x^2} + x^{2N+2}\right)y(x) = Ex^N y(x) \qquad (N = -1, 0, 1, \ldots). \tag{1.3}$$

Although the problem is posed here in one dimension, it often represents the radial equation for a problem in d-dimensions. For example, in the cases of Coulomb and harmonic oscillator problems, $m(m+1)/x^2$ might be the angular momentum term in d-dimensions with $m = \ell + (d-3)/2$. More generally, we might wish to consider generalizations of these potentials in which m is real and non-negative. Indeed, for the Pöschl–Teller family of potentials, m is simply a non-negative potential parameter.

The paper is organized as follows. In section 2, we review the asymptotic iteration method. This method was recently introduced [13] to construct exact solutions for a wide class of Schrödinger equations and in many cases provided excellent approximate results for non-trivial eigenvalue problems [13, 15]. In section 3, we investigate the exact solutions of the basic eigenvalue problem (1.1). We obtain analytic formulae for the eigenvalues of (1.1) and also explicit expressions for the eigenfunctions. In section 4, we study the exact solutions of the differential equation (1.3) where the connection with the Bender *et al* class is discussed in detail. In section 5, we developed the exact solutions for different classes of Pöschl–Teller potentials.

2. The asymptotic iteration method (AIM)

Consider the second-order homogeneous linear differential equation

$$y'' = \lambda_0(x)y' + s_0(x)y \tag{2.1}$$

for which $\lambda_0(x) \neq 0$ and $s_0(x)$ are functions in $C_\infty(a,b)$. In order to find a general solution to this differential equation, we rely on the symmetric structure of the right-hand side of (2.1). Differentiating (2.1) with respect to x, we find that

$$y''' = \lambda_1(x)y' + s_1(x)y, \tag{2.2}$$

where

$$\lambda_1 = \lambda_0' + s_0 + \lambda_0^2 \qquad \text{and} \qquad s_1 = s_0' + s_0\lambda_0.$$

Meanwhile the second derivative of (2.1) yields

$$y'''' = \lambda_2(x)y' + s_2(x)y \tag{2.3}$$

for which

$$\lambda_2 = \lambda_1' + s_1 + \lambda_0\lambda_1 \qquad \text{and} \qquad s_2 = s_1' + s_0\lambda_1.$$

Thus, for $(n+1)$th and $(n+2)$th derivatives, $n = 1, 2, \ldots$, we have

$$y^{(n+1)} = \lambda_{n-1}(x)y' + s_{n-1}(x)y \qquad \text{and} \qquad y^{(n+2)} = \lambda_n(x)y' + s_n(x)y \tag{2.4}$$

respectively, where

$$\lambda_n = \lambda_{n-1}' + s_{n-1} + \lambda_0\lambda_{n-1} \qquad \text{and} \qquad s_n = s_{n-1}' + s_0\lambda_{n-1}. \tag{2.5}$$

From the ratio of the $(n+2)$th and $(n+1)$th derivatives, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(y^{(n+1)}) = \frac{y^{(n+2)}}{y^{(n+1)}} = \frac{\lambda_n\left(y' + \frac{s_n}{\lambda_n}y\right)}{\lambda_{n-1}\left(y' + \frac{s_{n-1}}{\lambda_{n-1}}y\right)}. \tag{2.6}$$

We now introduce the 'asymptotic' aspect of the iteration method. For sufficiently large n, if

$$\frac{s_n}{\lambda_n} = \frac{s_{n-1}}{\lambda_{n-1}} := \alpha, \tag{2.7}$$

then (2.6) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln(y^{(n+1)}) = \frac{\lambda_n}{\lambda_{n-1}} \tag{2.8}$$

which yields the exact solution

$$y^{(n+1)}(x) = C_1\exp\left(\int^x \frac{\lambda_n(t)}{\lambda_{n-1}(t)}\,\mathrm{d}t\right) = C_1\lambda_{n-1}\exp\left(\int^x(\alpha + \lambda_0)\,\mathrm{d}t\right), \tag{2.9}$$

where C_1 is the integration constant, and the right-hand equation follows using (2.7) and (2.8). Substituting (2.9) into (2.4) we obtain the first-order differential equation

$$y' + \alpha y = C_1\exp\left(\int^x(\alpha + \lambda_0)\,\mathrm{d}t\right) \tag{2.10}$$

which, in turn, yields the general solution to (2.1) as

$$y(x) = \exp\left(-\int^x \alpha\,\mathrm{d}t\right)\left[C_2 + C_1\int^x \exp\left(\int^t(\lambda_0(\tau) + 2\alpha(\tau))\,\mathrm{d}\tau\right)\mathrm{d}t\right]. \tag{2.11}$$

3. An exactly solvable class of eigenvalue problem

In this section, we use AIM to investigate the second-order homogeneous linear differential equation

$$y'' = 2\left(axp(x) - \frac{(m+1)}{x}\right)y' - wp(x)y, \tag{3.1}$$

where a, m and w are real numbers. For special values of a and w, this differential equation can immediately be integrated. For example, if $w = 0$, then (3.1) yields

$$y(x) = C_1\int^x \frac{\mathrm{e}^{2a\int^t t'p(t')\mathrm{d}t'}}{t^{2m+2}}\,\mathrm{d}t + C_2.$$

Furthermore, if

$$p(x) = \mathrm{e}^{-x^2/w}\left[\frac{2(m+1)}{w}\int^x t^{-1}\,\mathrm{e}^{at^2/w}\mathrm{d}t + C_1\right],$$

then

$$y(x) = \mathrm{e}^{-\int^x wp(t)\mathrm{d}t}\left[C_1 \int^x \mathrm{e}^{\int^t wp(t')\mathrm{d}t'}\mathrm{d}t + C_2\right].$$

We are interested, however, in studying the exact solutions of (3.1) in which $p(x) = \frac{x^N}{1-bx^{N+2}}$, $N = -1, 0, 1, 2, 3, \ldots,$. In this case, (3.1) reads

$$y'' = 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(m+1)}{x}\right)y' - \frac{wx^N}{1-bx^{N+2}}y. \tag{3.2}$$

Denote

$$\lambda_0(x) = 2\left(\frac{ax^{N+1}}{1-bx^{N+2}} - \frac{(m+1)}{x}\right) \qquad \text{and} \qquad s_0(x) = -\frac{wx^N}{1-bx^{N+2}}, \tag{3.3}$$

we may then apply AIM and the asymptotic condition (2.7) yields for $n = 0, 1, 2, 3, \ldots$

- $w_n^m(-1) = n(2a + 2bm + (n+1)b)$ for $N = -1$
- $w_n^m(0) = 2n(2a + 2bm + (2n+1)b)$ for $N = 0$
- $w_n^m(1) = 3n(2a + 2bm + (3n+1)b)$ for $N = 1$
- $w_n^m(2) = 4n(2a + 2bm + (4n+1)b)$ for $N = 2$
- $w_n^m(3) = 5n(2a + 2bm + (5n+1)b)$ for $N = 3$
- ... etc.

Thus, by induction on N, we can easily verify that the $w_n^m(N)$ are given by

$$w_n^m(N) = b(N+2)^2 n(n+\rho), \qquad \rho = \frac{(2m+1)b + 2a}{(N+2)b}, \tag{3.4}$$

where $N = -1, 0, 1, 2, \ldots$ and $n = 0, 1, 2, 3, \ldots$. For the exact solutions $y_n(x)$, we use the generator of the exact solutions (2.11), namely

$$y_n(x) = C_2 \exp\left(-\int^x \alpha_n \,\mathrm{d}t\right), \tag{3.5}$$

where $n = 0, 1, 2, \ldots$ is the iteration step number. Direct computations, using $\lambda_k = \lambda'_{k-1} + s_{k-1} + \lambda_0\lambda_{k-1}$ and $s_k = s'_{k-1} + s_0\lambda_{k-1}$ where s_0 and λ_0 are given by (3.3), imply the following:

- $y_0(x) = 1$ since $w_0^m(N) = 0$ where $N = -1, 0, 1, 2, \ldots$
- $y_1(x) = -C_2(N+2)\sigma\left(1 - \frac{b(\rho+1)}{\sigma}x^{N+2}\right)$
- $y_2(x) = C_2(N+2)^2\sigma(\sigma+1)\left(1 - \frac{2b(\rho+2)}{\sigma}x^{N+2} + \frac{b^2(\rho+2)(\rho+3)}{\sigma(\sigma+1)}x^{2(N+2)}\right)$
- $y_3(x) = -C_2\frac{\sigma(\sigma+1)(\sigma+2)}{(N+2)^{-3}}\left(1 - \frac{3b(\rho+3)}{\sigma}x^{N+2} + \frac{3b^2(\rho+3)(\rho+4)}{\sigma(\sigma+1)}x^{2(N+2)} - \frac{b^3(\rho+3)(\rho+4)(\rho+5)}{\sigma(\sigma+1)(\sigma+2)}x^{3(N+2)}\right)$
- ... etc.

Consequently, we arrive at the following general formula for the exact solutions $y_n(x)$:

$$y_n(x) = (-1)^n C_2(N+2)^n(\sigma)_n\, {}_2F_1(-n, \rho+n; \sigma; bx^{N+2}), \tag{3.6}$$

where $(\sigma)_n = \frac{\Gamma(\sigma+n)}{\Gamma(\sigma)}$, $\sigma = \frac{2m+N+3}{N+2}$ and $\rho = \frac{(2m+1)b+2a}{(N+2)b}$. The Gauss hypergeometric function ${}_2F_1$ is defined by

$${}_2F_1(-n, b; c; z) = \sum_{k=0}^{n} \frac{(-n)_k(b)_k}{(c)_k} z^n, \tag{3.7}$$

that is to say, a polynomial of degree n in z.

4. A class of exactly solvable eigenvalue problems and their generalization

An important step in solving eigenvalue problems using AIM [13] is to find a suitable transformation that converts the eigenvalue problem under investigation into differential equation of form (2.1) with $\lambda_0 \neq 0$. As mentioned earlier (see [13] and [15]), the rate of convergence of AIM is influenced by this choice. The asymptotic solution of the eigenvalue problem usually provides a clear indication of a suitable transformation to use. For the class of eigenvalue problems of Bender *et al*

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{m(m+1)}{x^2} + a^2x^{2N+2}\right) y(x) = Ex^N y(x), \qquad 0 \leqslant x < \infty, \tag{4.1}$$

where $N = -1, 0, 1, 2, 3, \ldots, m \geqslant -1$, and $y(x)$ satisfying the *Dirichlet boundary condition* $y(0) = 0$, the asymptotic solutions of the differential equation (4.1), for x approaching 0 and ∞, suggest the following expression of the wavefunction:

$$y(x) = x^{m+1} \exp\left(-\frac{ax^{N+2}}{N+2}\right) f(x), \tag{4.2}$$

where $f(x)$ is now to be determined by AIM. Using (4.2) in (4.1), we find that the function $f(x)$ has to satisfy the differential equation

$$f'' = 2\left(ax^{N+1} - \frac{m+1}{x}\right) f' - wx^N f, \tag{4.3}$$

where $w = E - a(2m + N + 3)$. This differential equation is a special case of (3.2) with $b = 0$ and $x \in [0, \infty)$. Thus (3.4) yields

$$E = a(2n(N+2) + 2m + N + 3), \qquad n = 0, 1, 2 \ldots. \tag{4.4}$$

For the general solutions $f_n(x)$ ($n = 0, 1, 2, \ldots$), it is enough to take the limit in (3.6) as $b \to 0$. Thus, the solutions of (4.3) can be written using the limit relation

$$\lim_{b\to 0} {}_2F_1(-n, 1/b + a; c; zb) = {}_1F_1(-n; c; z), \tag{4.5}$$

in the form

$$f_n(x) = (-1)^n C_2 (N+2)^n (\sigma)_n\, {}_1F_1\left(-n, \sigma; \frac{2a}{N+2} x^{N+2}\right), \tag{4.6}$$

where $\sigma = \frac{2m+N+3}{N+2}$.

For $N = -1$, (4.1) represents a d-dimensional Schrödinger equation with the Coulomb potential, and the eigenvalues are $-a^2$, if m is the angular momentum parameter $m = \ell + (d-3)/2$ in d-dimensions. Replacing the parameter E (in this case E becomes a coupling parameter) by $E = Ze^2$ and $a^2 = -E_{nm}$, we easily recover, from (4.4), the well-known exact eigenvalues $E_{nm} = -\frac{Z^2e^4}{4(n+m+1)^2}$ for the Coulomb problem in d dimensions. Meanwhile, the case $N = 0$ represents the d-dimensional Schrödinger equation for the harmonic oscillator potential and the parameter E itself is now the eigenvalue $E = E_{nm}$, where m is again the angular momentum parameter $m = \ell + (d-3)/2$. In this case, (4.4) yields the well-known exact eigenvalues $E_{nm} = a(4n + 2m + 3)$ for this harmonic oscillator problem in d dimensions. The parameter m may be more generally any non-negative real number. For $N > 0$ we emphasize that the eigenvalues of the boundary-value problem (4.1) do not correspond to those of a conventional Schrödinger equation. However, the zero eigenvalues may be the ground-state energies corresponding to an effective potential

$$V(x) = \frac{m(m+1)}{x^2} + a^2x^{2N+2} - gx^N, \qquad (g > 0).$$

Indeed, our method gives us the conditions on m, a and g under which such zero eigenvalues might occur. For example, for $N = 2$, we have for

$$\left(-\frac{d^2}{dx^2} + \frac{m(m+1)}{x^2} + a^2x^6 - gx^2\right) y(x) = Ey(x), \qquad 0 \leqslant x < \infty \quad (4.7)$$

the energy is zero $E = 0$ if

$$g = a(8n + 2m + 5), \qquad n = 0, 1, 2, \ldots. \quad (4.8)$$

5. Pöschl–Teller potentials

In this section, we show how the asymptotic iteration method can be used to generate the exact solutions of the different families of Pöschl–Teller potentials [16–18]

$$V_{\mathrm{I}}(u) = k^2\left(\frac{\alpha(\alpha+1)}{\cos^2(ku)} + \frac{\beta(\beta+1)}{\sin^2(ku)}\right) \qquad \left(0 < ku < \frac{\pi}{2}, \alpha, \beta > 0\right) \quad (5.1)$$

and

$$V_{\mathrm{II}}(u) = k^2\left(\frac{\beta(\beta-1)}{\sinh^2(ku)} - \frac{\alpha(\alpha+1)}{\cosh^2(ku)}\right) \qquad (0 < ku < \infty, \alpha > \beta)\,. \quad (5.2)$$

Our approach shows that the exact solutions of Pöschl–Teller potentials follow directly from solutions of the differential equation (3.2) through elementary transformations. The main idea is to use trigonometric or hyperbolic mappings that make the potentials (5.1) and (5.2) rational and then make the corresponding Schrödinger equation suitable to be analysed with AIM.

5.1. The first class of Pöschl–Teller potentials

We consider the one-dimensional Schrödinger equation of a quantum particle trapped by the Pöschl–Teller potential $V_{\mathrm{I}}(u)$

$$-\frac{d^2y}{du^2} + V_{\mathrm{I}}(u)y = \mathcal{E}y, \quad (5.3)$$

where $V_{\mathrm{I}}(u)$ is given by (5.1). The potential $V_{\mathrm{I}}(u)$ is closely related to several other potentials which are widely used in molecular and solid state physics like the symmetric Pöschl–Teller potential $\alpha = \beta \geqslant 0$, and the Scarf potential $-\frac{1}{2} \leqslant \alpha < 0$. The substitution $x = ku$, $0 < x < \frac{\pi}{2}$, yields

$$-\frac{d^2y}{dx^2} + \left(\frac{\alpha(\alpha+1)}{\cos^2(x)} + \frac{\beta(\beta+1)}{\sin^2(x)}\right) y = Ey, \qquad \left(E = \frac{\mathcal{E}}{k^2}\right). \quad (5.4)$$

The potential $V_{\mathrm{I}}(x) = \alpha(\alpha+1)/\cos^2(x) + \beta(\beta+1)/\sin^2(x)$ is periodic. However, each period is separated from the next by an infinite potential barrier so it can be studied within one period, say $0 < x < \frac{\pi}{2}$. Clearly, (5.4) is a smooth approximation for $\alpha, \beta \to 0^+$ of the infinite square-well potentials over the interval $\left[0, \frac{\pi}{2}\right]$. In order to apply the asymptotic iteration method, discussed earlier, to solve the eigenvalue problem (5.4), we have to transform it into a differential equation of form (2.1) with suitable $\lambda_0 \neq 0$. The asymptotic solutions of (5.4) as x approaches 0^+ and $\frac{\pi}{2}^-$ suggest for $y(x)$ the following expression:

$$y(x) = \cos^{\alpha+1}(x)\sin^{\beta+1}(x) f(x), \quad (5.5)$$

where $f(x)$ is found by means of AIM. After substituting (5.5) into (5.4), we obtain the following differential equation for f:

$$f'' = 2\left((\alpha+1)\tan(x) - (\beta+1)\cot(x)\right) f' - wf, \quad (5.6)$$

where $w = E - (\alpha + \beta + 2)^2$. Further, if we use the substitution $t = \cos(x), 0 < t < 1$, we arrive at the following equation:

$$f'' = 2\left(\frac{\left(\beta + \frac{3}{2}\right)t}{1 - t^2} - \frac{\alpha + 1}{t}\right) f' - \frac{w}{1 - t^2} f, \qquad f' = \frac{\mathrm{d}f}{\mathrm{d}t}, \tag{5.7}$$

which we may compare with (3.2) for $b = 1, a = \beta + \frac{3}{2}, m = \alpha$ and $N = 0$. Thus from (3.4) we have $w = 4n(\alpha + \beta + n + 2)$ where $n = 0, 1, 2 \ldots$. Finally, the eigenvalues for equation (5.4) are given by

$$E_n = (\alpha + \beta + 2 + 2n)^2, \qquad n = 0, 1, 2, \ldots. \tag{5.8}$$

Directly solutions $f_n(t)$ for the eigenvalue problem (5.8) can now be obtained from (3.6) with the substitution $b = 1$ and $N = 0$. Thus, we have

$$f_n(t) = (-2)^n C_2(\sigma)_n\, {}_2F_1(-n, \rho + n, \sigma; t^2), \qquad t = \cos(x), \tag{5.9}$$

where $\rho = \alpha + \beta + 2$ and $\sigma = \alpha + \frac{3}{2}$. Consequently, the wavefunctions for the first Pöschl–Teller potential (5.4) are

$$\begin{aligned} y_n(x) &= (-2)^n C_2 \left(\alpha + \tfrac{3}{2}\right)_n \cos^{\alpha+1}(x) \sin^{\beta+1}(x)\, {}_2F_1\left(-n, \alpha + \beta + 2 + n, \alpha + \tfrac{3}{2}; \cos^2(x)\right) \\ &= (-2)^n C_2 n! \cos^{\alpha+1}(x) \sin^{\beta+1}(x) P_n^{(\alpha+\frac{1}{2}, \beta+\frac{1}{2})}(1 + 2\cos^2 x) \end{aligned} \tag{5.10}$$

using the relation [19] between the hypergeometric function and Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ of degree n. The constant C_2 in (5.10) is still to be determined by normalization.

5.2. The second class of Pöschl–Teller potentials

We now consider the Schrödinger equation

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}u^2} + V_{\mathrm{II}}(u) y = \mathcal{E} y, \qquad 0 < u < \infty, \tag{5.11}$$

where $V_{\mathrm{II}}(u)$ is given by (5.2). With the substitution, $x = ku$, we have

$$-\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\beta(\beta - 1)}{\sinh^2(x)} - \frac{\alpha(\alpha + 1)}{\cosh^2(x)}\right) y = E y, \qquad 0 < x < \infty. \tag{5.12}$$

We may assume $\alpha > \beta$, for if $\alpha < \beta$, we can change $\alpha \to -\alpha - 1$, because the equation remains unchanged under $\alpha \to -\alpha - 1$ and $\beta \to -\beta + 1$. Because of the asymptotic solutions for y as x approaches 0^+ and ∞, we may assume the exact form of the wavefunction $y(x)$ to be

$$y(x) = \cosh^{-\alpha}(x) \sinh^{\beta}(x) f(x), \tag{5.13}$$

where the function $f(x)$ is to be found with AIM. After substituting (5.13) into (5.12), we obtain the following differential equation for $f(x)$:

$$f'' = 2\left(\alpha \tanh(x) - \beta \coth(x)\right) f' - w f, \tag{5.14}$$

where $w = E + (\alpha - \beta)^2$. Further, by means of the substitution $t = \sinh(x)$, we can easily obtain

$$f'' = 2\left(\frac{\left(\alpha - \frac{1}{2}\right)t}{1 + t^2} - \frac{\beta}{t}\right) f' - \frac{w}{1 + t^2} f, \tag{5.15}$$

where the prime refers to the derivative with respect to the variable t. This differential equation can be compared with (3.2) with $b = -1, a = \alpha - \frac{1}{2}, m = \beta - 1$ and $N = 0$. Thus, using

(3.4), we find the following expression for w: $w = 4n(\alpha - \beta - n)$, where $n = 0, 1, 2 \ldots$. Finally, the eigenvalues for (5.12) are given explicitly by the expression

$$E_n = -(\alpha - \beta - 2n)^2, \qquad n = 0, 1, 2, \ldots < (\alpha - \beta)/2. \tag{5.16}$$

Consequently, the expected maximum number of quanta is

$$n_{\max} = [(\alpha - \beta)/2], \tag{5.17}$$

where $[(\alpha - \beta)/2]$ stands for the closest integer to $(\alpha - \beta)/2$ that is smaller than $(\alpha - \beta)/2$.

With the substitution $b = -1$ and $N = 0$, we find using (3.6) that the exact solutions of $f_n(y)$ which satisfy (5.14) for $n = 0, 1, 2, \ldots < (\alpha - \beta)/2$ are

$$\begin{aligned} f_n(y) &= (-2)^n C_2 \left(\beta + \tfrac{1}{2}\right)_n {}_2F_1\left(-n, \beta - \alpha + n; \beta + \tfrac{1}{2}; -t^2\right) \\ &= (-2)^n C_2\, n!\, P_n^{(\alpha-\frac{1}{2}, \beta-\frac{1}{2})}(1 + 2t^2), \end{aligned} \tag{5.18}$$

where in the last line we have expressed the hypergeometric function in terms of the Jacobi polynomial in order to make easier the comparison with the results found in the literature [20]. Up to a normalization constant, the wavefunction $y(x)$ of (5.2) reads

$$y_n(x) = (-2)^n C_2\big(\beta + \tfrac{1}{2}\big)_n \cosh^{-\alpha}(x) \sinh^{\beta}(x)\, {}_2F_1\left(-n, \beta - \alpha + n; \beta + \tfrac{1}{2}; -\sinh^2(x)\right). \tag{5.19}$$

6. Conclusion

In this paper, we have applied the asymptotic iteration method to obtain the exact solutions for an interesting class of differential equations (3.2). We have shown that the exact solutions for the class of problems studied by Bender *et al* are obtained by an application of AIM to a transformation of (3.2). This allows us to obtain the exact solutions for an extended class of eigenvalue problems (4.1) with $m \geqslant -1$. Complete solutions for the well-known Coulomb and harmonic oscillator potentials follow directly by setting $N = -1, 0$ respectively in (4.1). By means of trigonometric and hyperbolic coordinate transformations of t, in terms of which the first and second kinds of Pöschl–Teller potentials become rational functions of t, AIM also provides a direct way of generating the exact solutions to the Schrödinger eigenvalue problem generated by these potentials.

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